APPLICATIONS OF SYNCHRONIZATION IN SOUND SYNTHESIS

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ABSTRACT

The synchronization of natural and technical periodic processes can be simulated with self-sustained oscillators. Under certain conditions, these oscillators adjust their frequency and their phase to a master oscillator or to other self-sustained oscillators. These processes can be used in sound synthesis for the tuning of non-linear oscillators, for the adjustment of the pitches of other oscillators, for the synchronization of periodic changes of any sound parameters and for the synchronization of rhythms. This paper gives a short introduction to the theory of synchronization [1, 2, 3, 4, 5], shows how to implement the differential equations which describe the self-sustained oscillators and gives some examples of musical applications. The examples are programmed as mxj~ externals for MaxMSP. The Java code samples are taken from the perform routine of these externals. The externals and Max patches can be downloaded from http://www.icst.net/downloads.

1. INTRODUCTION

Temporally coordinated processes are said to be synchronized. If the processes are periodic, their frequencies and phases can be coordinated. The spontaneous synchronization of machines was first described by Christiaan Huyghens (1629-1695), who observed that clocks affixed to the same support can synchronize themselves. Synchronization also plays an important role in laser technology, in neuronal nets, in chemical reactions, etc. Synchronization can be forced by any of a variety of impulse generators. More interesting is the synchronization of several non-hierarchically organized systems, which can happen with non-linear systems. Examples of synchronization by an external force are the control of cardiac activity by a pace maker and the synchronization of biological cycles through circadian rhythm. An example for the mutual synchronization of oscillating systems is the synchronization of the coordinated clapping of an audience. These systems have in common that they are not linear and that they oscillate without external excitation. They are called self-sustained oscillators.

Nonlinear and chaotic oscillators have yet often been used in sound synthesis and control since the nineties [6, 7, 8] and some of them have been implemented as unit generators (UGens) in sound synthesis languages [9] but only few literature exists on coupled nonlinear oscillators [10] and their synchronization [11].

2. SELF-SUSTAINED OSCILLATORS

Self-sustained oscillators have a natural frequency and compensate for energy loss by an inner energy source. The trajectory of the oscillation in phase space (x, $\dot{x}$) is a stable limit cycle. We first consider the behavior of a self-sustained oscillator from a purely qualitative point of view. Figure 1 shows the simplest limit cycle: a circle. The state of the unperturbed oscillator is described by a point rotating along the limit circle. In a coordinate system rotating with the same angular velocity as the unperturbed oscillator, the system's state can be described as a stationary point. If the oscillator is perturbed, for example moved from state 1 to state 2, the influence of the attractor (here the circle) gradually dissipates the amplitude change, but the phase shift remains (state 3). The fact that very weak external forces can perturb the phase is one of the main reasons why self-sustaining oscillators can synchronize themselves.

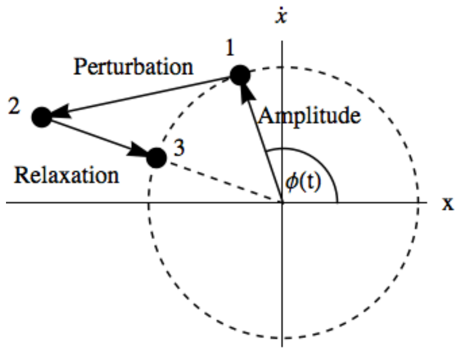

Figure 1. The influence of the attractor (here the circle) of a self-sustained oscillator gradually dissipates the amplitude change of a perturbation, but the phase shift remains.

2.1 The Van der Pol Oscillator

While attempting to explain the non-linear dynamics of vacuum tube circuits, the Dutch electrical engineer Balthasar van der Pol derived the equation

$$\ddot{x} = -\omega^2 x + \mu(1 - x^2)\dot{x} \tag{1}$$

The equation describes a linear oscillator $\ddot{x} = -\omega^2 x$ with an additional non-linear term $\mu(1 - x^2)v$. When $|x| > 1$, negative damping results, which means that energy is introduced into the system. For the following calculation we write the differential equation above as a system of two equations

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\omega^2 x + \mu(1 - x^2)v \end{aligned} \tag{2}$$

Figure 2 shows solutions of the equation for given initial values. The limit cycle is quickly reached, regardless of the initial values. As μ increases, the limit cycle becomes more and more deformed.

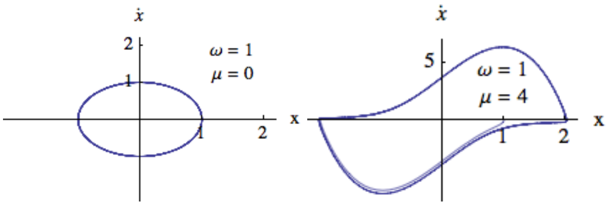

Figure 2. The limit cycle of the Van der Pol oscillator for different values of the parameter μ.

Figure 3 shows the spectrum of the oscillation for two values of μ.

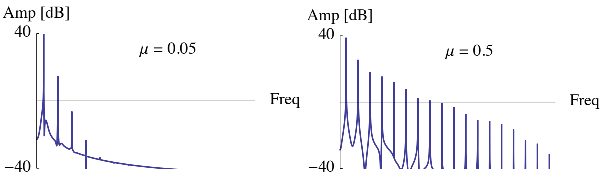

Figure 3. The spectrum of the Van der Pol oscillator becomes richer with growing parameter μ.

2.2 The Rössler Oscillator

The following homogeneous system of two linear and one non-linear differential equations can provide chaotic behavior and is named after its discoverer Otto E. Rössler:

$$\begin{aligned} \dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + xz - cz \end{aligned} \tag{3}$$

The phase space is three-dimensional. When z is small, the trajectory is close to the x-y plane and the approximations $\dot{x} = -y$ and $\dot{y} = x$ hold. Hence it follows that $\ddot{x} = a\dot{x} - x$. This equation describes an oscillation with negative damping which in the phase space is a spiral moving outward from the center. When x becomes larger than c, the third equation causes z to increase exponentially, causing the trajectory to rise quickly out of the x-y plane (see Figure 4). A large value for z makes $\dot{x}$ negative, so that x becomes smaller than c and the trajectory descends to the x-y plane again.

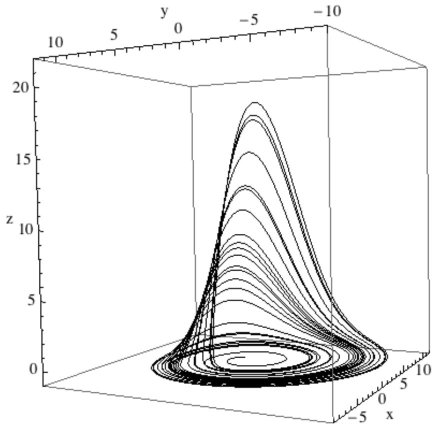

Figure 4. Chaotic trajectory of the Rössler system in state space (x, y, z).

2.3 Implementation

There are various methods to get discrete systems, that is difference equations from differential equations, for example Euler's Method, the improved Euler's Method (or Heun's Method) or the Runge - Kutta Method. The last one is often used since it provides good results even for rather large time steps (see the SuperCollider implementations of nonlinear oscillators in [7]). With every method the results can be improved shortening the time step. In order to keep the code short in the presented MaxMSP externals we use Euler's Method and the sample period as time step. In order to implement the Van der Pol Oscillator (mxj~ *smc_v_d_pol_1*) we first calculate the acceleration a according to the differential equation above (eq. 2). Then we increment the velocity by the acceleration times dt and displacement x by velocity times dt. Taking the sample period (for sound) or frame period (for computer animations) as the time unit ($dt = 1$), we get:

```
a = (-c*x + mu*(1 - x*x)*v);
v += a;
x += (v+in[i]);
```

Where $c = \omega^2$ and in[i] is the i-th sample of an excitation input buffer. Since in the Rössler system x, y and z depend on each other, we store copies of the actual values:

```
x = x0; y = y0; z = z0;
x += dt*(- y0 - z0);
y += dt*(x0 + a*y0);
z += dt*(b + (x0  - c)*z + in[i]);
```

3. SYNCHRONIZATION

3.1 Synchronization Using Periodic Excitation

First we describe a quasi-linear oscillator, that is, an oscillator with very little non-linearity. Let this non-linearity be an arbitrary function $n()$ of the state of the oscillator (x, $\dot{x}$).

$$\ddot{x} = -\omega_0^2 x + n(x,\dot{x}) \tag{4}$$

The solution of the equation without the non-linear term is

$$x(t) = A\sin(\omega_0 t + \phi_0) \tag{5}$$

If we excite the oscillator with the periodic force $f(t) = \varepsilon\cos(\omega t + \phi_0^e)$, we obtain the equation

$$\ddot{x} = -\omega_0^2 x + n(x,\dot{x}) + f(t) \tag{6}$$

The instantaneous phase of the exciting force is $\phi_e = \omega t + \phi_0^e$. The frequency ω generally differs from the frequency ω_0 of the autonomous oscillator. The difference of the two frequencies ω - ω_0 is called detuning. Because a perturbance of amplitude decay rapidly, it suffices to consider the behavior of the phase. The limit cycle of the quasi-linear oscillator is a circle on which the point representing the phase rotates with the frequency ω_0. In a coordinate system revolving with the frequency ω of the exciting force, the phase point of the oscillator has the angular velocity ϕ - ϕ_e (Figure 5 left). The exciting force is represented as a vector of length ε (dotted arrows in Figure 5 right), acting at the angle $\phi^a = \phi_0^e + \pi/2$. The effect of the force depends on the phase difference ϕ - ϕ_e. At the points 1 and 2, the force acts perpendicularly to the trajectory and hence does not affect the phase. At all other points a force results that drives the points toward point 1. Hence point 1 is a stable fixed point, point 2 an unstable fixed point.

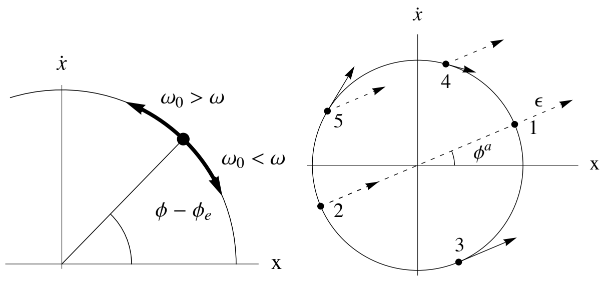

Figure 5. Coordinate system revolving with the frequency ω of the exciting force. The exciting force is represented as a constant vector. The effect of the force depends on the phase difference ϕ - ϕ_e.

If the detuning is zero, any initial phase difference between the excitation and the quasi-linear oscillator is reduced until ϕ = ϕ_e -ϕ^a and phase locking obtains between the two. If the detuning increases (e.g. when ω_0 > ω), then two tendencies are in competition with each other: rotation (solid arrows in Figure 6 left) and the force of the excitation. The phase difference between excitation and oscillator levels off at a certain value $\Delta\phi$ (function 1 in Figure 6 right). Their movements are synchronous but not identical. If the detuning is large, the force offers insufficient resistance to the rotation. The result is a function $\Delta\phi(t)$ that remains constant for a certain time and then slips to make a quick full rotation (function 2 in Figure 6 right). The phase point starts to rotate with the so-called beat frequency Ω.

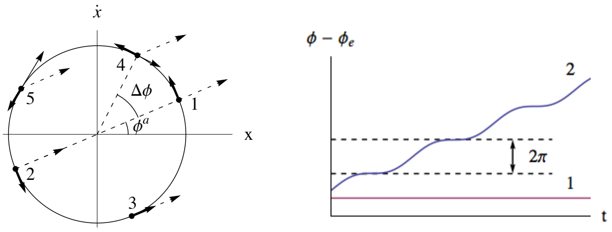

Figure 6. State space of a periodically excited quasi-linear oscillator (left figure). Phase difference ϕ - ϕ_e versus time for a asynchronous (2) and for a synchronous (1) state (right figure).

3.2 Mutual Synchronization of Coupled Oscillators

Let us first consider two coupled limit-cycle oscillators. If we represent them as a system of first-order differential equations, we can write:

$$\begin{aligned} \dot{x}_1 &= F_1(x_1) + \varepsilon P_1(x_1, x_2) \\ \dot{x}_2 &= F_2(x_2) + \varepsilon P_2(x_1, x_2) \end{aligned} \tag{7}$$

Here the x_i are vectors, F_i and P_i are arbitrary functions and $|\varepsilon|$<<1. If the natural frequencies of the oscillators ω_i are approximately equal the behavior of the oscillators can be described by the following equations for their phases ϕ_i.

$$\begin{aligned} \dot{\phi}_1 &= \omega_1 + \varepsilon q_1(\phi_2 - \phi_1) \\ \dot{\phi}_2 &= \omega_2 + \varepsilon q_2(\phi_1 - \phi_2). \end{aligned} \tag{8}$$

The q_i are 2π-periodic functions. Then for the difference of the two phases $\theta = \phi_2 - \phi_1$ we have:

$$\dot{\theta} = \Delta - 2\varepsilon q(\theta) \tag{9}$$

where $\Delta = \omega_2 - \omega_1$ and $q(\theta) = q_2(\theta) - q_1(\theta)$. For there to be synchronization, the phase difference θ must be constant, that is $\dot{\theta} = 0$. Hence we have

$$q(\theta) = \Delta / 2\varepsilon\,. \tag{10}$$

For the simplest 2π-periodic function, the sine wave, we obtain the so-called Adler equation [1] of the first degree $\dot{\theta} = \Delta - 2\varepsilon\sin(\theta)$. $\dot{\theta}$ can only be made to disappear when $|\Delta| < 2\varepsilon$. If $|\Delta|$ becomes greater than $|2\varepsilon|$, the coupled system begins to beat. The beat frequency Ω can be calculated from the equation $d\theta/dt = \Delta - 2\varepsilon q(\theta)$. By solving for dt and integrating dt over one period of θ, we obtain the period's duration and from it the beat frequency:

$$\Omega = 2\pi\left(\int_0^{2\pi} \frac{1}{2\varepsilon q(\theta) - \Delta} d\theta\right)^{-1}. \tag{11}$$

For $q(\theta) = \sin(\theta)$ we obtain

$$\Omega = \sqrt{\Delta^2 - 4\varepsilon^2}\,, \tag{12}$$

which is the same function as for the synchronization of an oscillator using periodic excitation.

4. APPLICATIONS

4.1 Single Oscillators

Sounds having rich and amplitude-dependent spectra can be produced with non-linear oscillators. The fundamental frequency of non-linear oscillators can depend on the amplitude. Figure 3 shows the spectrum of a Van der Pol oscillator for different values of the parameter μ. With increasing μ the spectrum becomes richer and simultaneously the frequency decreases. Within a broad range of the parameter μ, the frequency can be controlled by a periodic excitation. Such an oscillator can also be integrated into the Van der Pol oscillator.

The Max patch *smc_v_d_pol_1* represents a Van der Pol oscillator with a natural frequency of ω_0 and a non-linearity factor of μ. It can be excited by a sine wave of frequency ω and amplitude ε. The range of ω within which the oscillator is synchronized to the exciting frequency increases as μ and ε increase. The variation of the phase difference between excitation and oscillation, as well as the transitions between synchronous, beating and asynchronous behaviors, can be visualized by showing the sum of the excitation and the oscillation signals in a phase diagram. The screenshots of the Max patch in Figure 7 show to the upper left the waveform of the Van der Pol oscillator, to the lower left that of the excitation (amplified) and to the right the phase diagram of their sum. For these figures, the same values were always used for ω_0, μ and ε. Comparing the figures a) and b), one sees that the oscillator adopts the exciting frequency ω within a large frequency range. When the frequency is low a), the phases of the two waves are nearly the same. Hence there is a large deflection along the x-axis in the phase diagram showing the sum of the waveforms. When the frequency is high, the phases are nearly inverted (b) and the phase diagram shows only a small deflection. The figure c) shows the transition to asynchronous behavior. If the proportion between the natural frequency of the oscillator ω_0 and the excitation frequency ω is approximately simple, then within a certain range the frequency of the Van der Pol oscillator is synchronized so that $\omega/\omega_0 = m/n$ (with integral n and m) [5]. Here one speaks of higher order synchronization.

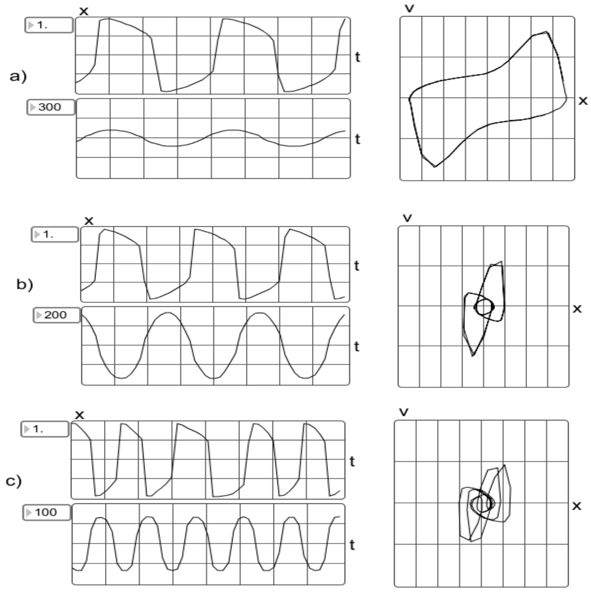

Figure 7. Waveforms of the Van der Pol oscillator (upper left) and of the excitation (lower left) and phase diagram of their sum (to the right) for different ratios between the natural frequency of the oscillator ω_0 and the excitation frequency ω.

Another possibility for controlling the frequency is to measure the frequency produced and to change the constant c until it matches the given frequency. For oscillators producing waveforms with only two zero crossings per period, we can easily measure the frequency by counting the sample periods between every second zero crossing. The following code sample is from the mxj external *smc_v_d_pol_2*, where *pcount* is a counter for the period, *dc* a correcting summand for the coefficient *c*, *p* the given period and *dcf* a factor for the speed with which the constant *c* is adjusted.

```
if(xd1*x >= 0) pcount += 1;
else{
      p = pcount + 1; pcount = 0;
      dc -= dcf*(per - 2.f*p/44100.f); }
```

Both methods result in a certain naturalness of the sound produced. Synchronization with an oscillator causes beatings and abrupt transitions to chaotic oscillations when the frequency of the oscillator differs too much from the eigenfrequency of the Van der Pol oscillator. Controlling the frequency gives rise to continuous changes in pitch like portamento.

4.2 Mutual Synchronization of Coupled Oscillators

Only in the simplest cases can we treat the behavior of several coupled oscillators analytically [3]. Therefore, in what follows we will only describe the qualitative behavior of arrays of coupled oscillators and provide Max patches for experimentation. In the patch *smc_vdp_lin_array* N oscillators are generated by the mxj~ object *smc_vdp_lin_array* and arranged in a circle. The frequencies ω_0 and the non-linearity factors μ_0 of the individual uncoupled oscillators are uniformly randomly distributed within a range chosen by the user. An oscillator is coupled to its two neighbors by using a part of the sum of their velocities as its excitation. The following code from the mxj~ object shows how the velocity v and amplitude x are calculated for the oscillators 1 to n-2. The oscillators 0 and n-1 at the beginning and end of the array are treated separately.

```
for(int k = 1; k < n-1; k++){
  v[k] += (-c[k]*x[k]
           + mu[k]*(1 -x[k]*x[k])*v[k]
           + fb*([k-1] + v[k+1]));
  x[k] += (v[k] + in[i]);}
```

The behavior of the coupled oscillators is easy to describe for extreme values of ε. When ε is zero, the oscillators are independent and oscillate at their natural frequencies. When ε exceeds a certain value, the oscillators are in synchrony. The transition from complete synchrony to asynchrony as ε diminishes takes place through bifurcation. Figure 8 shows the frequencies of five Van der Pol oscillators as a function of their feedback factors. In this simulation over 25000 samples, the feedback is realized in the same way as in the Max patch. Over the duration of the simulation the feedback factor *fb* sinks from 0.0067 to 0.

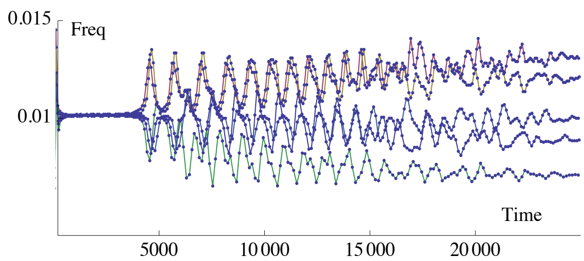

Figure 8. The transition from complete synchrony to asynchrony (for diminishing ε) takes place through bifurcation. The illustration shows the frequencies of five coupled Van der Pol oscillators.

4.3 Synchronizing of Chaotic Oscillators

Coupled chaotic oscillators like the Rössler oscillator (Section 2.2) can synchronize after just a few steps, even though the resulting time series is chaotic. This state is known as complete synchronization [4]. The code example below is taken from the mxj~ object *smc_roessler*, which is used in the following patches. The oscillation frequency depends essentially on the constant *dt*. In the examples which follow, $a = b = 0.2$ and the constant c is variable. When $c < 3$, periodic oscillation results, when $c > 3$, the periods are doubled, leading to chaotic behavior when $c \sim 4.3$. The oscillator is coupled with an external oscillator by adding the output of the latter to the velocity in z-direction.

```
x += dt*(-y0 - z0);
y += dt*(x0 - a*y0);
z += dt*(b + (x0 - c)*z + in[i]);
```

Figure 9 (Max patch *smc_roessler_1*) shows how the Rössler oscillator, within a certain range, takes on the frequency of an exciting oscillation. At the same time, it demonstrates how the influence of excitation can change originally chaotic behavior (a) into periodic oscillation (b). The figures show at the top left the oscillation of the Rössler oscillator, at the lower left the excitation (considerably enlarged) and on the right the trajectory in the phase space.

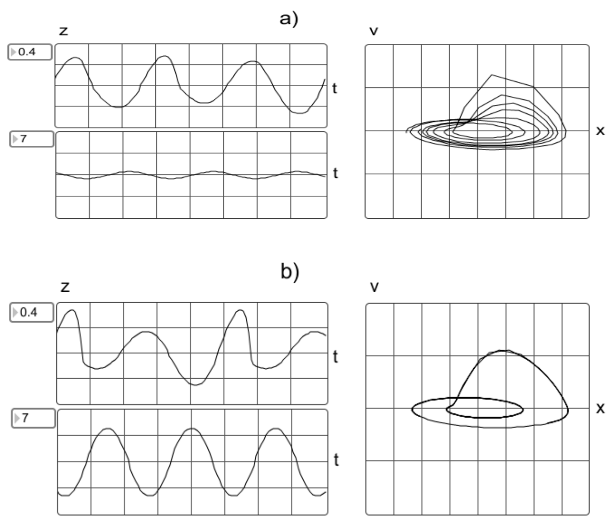

Figure 9. Waveform of the oscillation of the Rössler system (to the upper left), waveform of the excitation (at the lower left) and phase diagram of the Rössler system. Chaotic behavior a), periodic oscillation a) .

The Max patch *smc_roessler_2* demonstrates how coupling two Rössler oscillators having nearly identical frequencies leads to synchronization. If the oscillators' parameters are the same, the synchronization can be perfect.

Even the smallest differences in the initial conditions of chaotic systems lead rapidly to different trajectories. So it is astonishing that two identical uncoupled systems can be synchronized by being excited with noise. Chaotic systems can "forget", as it were, their initial conditions (Max patch *smc_roessler_3*). Depending on the values of ω and μ, it can take a long time for the synchronization to become perfect. We can see the same behavior in non-chaotic non-linear systems. Figure 10 shows the time series of two uncoupled Van der Pol oscillators ($\omega = 0.1$, $\mu = 0.25$) excited by white noise.

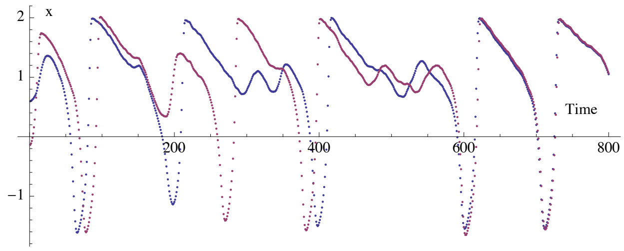

Figure 10. Time series of two uncoupled Van der Pol oscillators excited by noise.

4.4 Synchronizing Rhythms

Within one period of oscillation, many self-sustained oscillators go through a phase of slow variation and a phase of rapid variation. For example, neurons slowly build up tension and then discharge it rapidly. Oscillators of this type are called integrate-and-fire or accumulate-and-fire oscillators. Examples are the Van der Pol oscillator with a large nonlinearity or the so-called skew tent map. The mxj~ object *smc_integrate_fire* realizes a simple oscillator by incrementing the variable x by a constant c and a random value until x is greater than 1. Then x is reset to zero, the feedback variable *sync* is set to 1 and the discharge is indicated by a "bang". The value of the variable *sync* is quickly reduced.

```
x += (c + drand*Math.random());
if(x + exc > 1)
{   sync = 1.f; x = 0; outletBang(2);}
outlet(1,sync); sync*=0.9; outlet(0,x);
```

Several such objects are coupled together in the Max patch *smc_integrate_and_fire*. They produce more or less synchronous rhythms, depending on the parameter values used.

4.5 Synchronizing Any Parameter

Sine waves normally fuse to make a single sound if their frequencies are harmonic. But if they sound from different directions, do not begin at the same time or have different vibratos, they do not fuse. In order to synchronize their vibratos, not only the amplitudes and frequencies of the vibratos must be identical, but also their relative (i.e. wrapped) phases. It is difficult to measure and control these parameters if independent oscillators are used. Even if we can adjust the frequencies, the phases normally differ. Using mutually coupled self-sustained oscillators to produce the frequencies of the vibratos we can let them synchronize just by adjust the parameter which controls the mutual coupling.

5. CONCLUSIONS

Self-sustained oscillators not only can be used to produce interesting sounds but also as a means to control parameters. Their capability to synchronize can be used to control processes which we cannot or do not want to control externally.

6. REFERENCES

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