An algorithm for real-time harmonic microtuning

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Abstract — **Subtle inflections of pitch, often performed intuitively by musicians, create a harmonically sensitive expressive intonation. As each new pitch is added to a simultaneously sounding structure, very small variations in its tuning have a substantial impact on overall harmonic comprehensibility.**

In this project, James Tenney's multidimensional lattice model of intervals ('harmonic space') and a related measure of relative consonance ('harmonic distance') are used to evaluate and optimize the clarity of sound combinations. A set of tuneable intervals, expressed as whole-number frequency ratios, forms the basis for real-time harmonic microtuning. An algorithm, which references this set, allows a computer music instrument to adjust the intonation of input frequencies based on previously sounded frequencies and several user-specified parameters (initial reference pitch, tolerance range, pitch-class scaling, prime limit).

Various applications of the algorithm are envisioned: to find relationships within components of a spectral analysis, to dynamically adjust a computer instrument to other musicians in real time, to research the tuneability of complex microtonal pitch structures. More generally, it furthers research into the processes underlying harmonic perception, and how these may lead to musical applications.

I. Introduction

In *A History of 'Consonance' and Dissonance'* James Tenney identifies several distinct conceptions of these two terms in the theory and practice of Western music. In particular, he singles out contributions made by Hermann von Helmholtz in identifying a potential *psychoacoustic* basis for defining them: namely, as properties of sound that have to do with the beating speeds of partials and combination tones in a complex structure of pitches. Helmholtz posits a relationship between consonance and the elimination of slow beats caused by unisons of common partials. Conversely, he associates dissonance with maximal roughness (partials beating between 30 and 40 Hz). Helmholtz' theory suggests that consonance, as well as *dissonance,* can be most effectively maximized by tuning sounds in Just Intonation, that is, as integer ratios of frequencies, because such sounds have a repeating (periodic) structure. Periodicity emphasizes both the sensation of stability and smooth fusion in beatless consonances as well as the regularity of intermittent pulsations causing roughness in dissonance.

Composers in the 20th century have established a basis for *musically* exploring Helmholtz' premises about harmonic perception. Arnold Schoenberg, John Cage and Edgard Varèse, among numerous others, contributed to the emancipation of dissonance and noise (inharmonic sound complexes) as acceptable musical material. Harry Partch, Ben Johnston and La Monte Young extended the late 15th Century model of Just Intonation, which was a tuning system using small-number frequency ratio intervals derived from the prime numbers 2, 3, and 5, to include relationships produced by many higher prime numbers (7, 11, 13, 17, 19, 23, 29, 31, ...).

Building on Leonhard Euler's theory of mathematically evaluating relative consonance, Tenney proposes a general multidimensional lattice model called harmonic space, in which the *harmonic distance* (HD) between two pitches comprising a musical interval may be well defined. Each interval is represented in this space by the prime factor exponents of a frequency ratio b/a, expressed in lowest terms, and measured from an arbitrarily tuned origin (1/1). This ratio (b/a) may be exactly equal to the initial interval, or it may be a nearby approximation, taking into account detuning within a specified range of *tolerance.* Harmonic distance (HD) is defined by the following equation.

$$HD = \log(ab) \qquad (1)$$

One may also simply consider the lowest-terms product of numerator and denominator – the integer (ab) – which I call HD-product.

Thus, the harmonic space representation of musical intervals may be constrained by *tolerance*, by the number of prime factors included in the model (dimensionality or *prime limit),* and finally by *harmonic distance.*

II. Melodic and Harmonic Relationships

Aristoxenus, writing in the fourth century B.C., distinguishes the continuously gliding pitch changes characteristic of *speech* from the sustained, discrete tones used in *singing.* In particular, he notes that the more precisely such tones are maintained at correctly chosen pitch-heights, the more clarity melody acquires.

In the simplest sense, melody is based on three possible relations between pitches: *the same, higher* and *lower*. The property of being the 'same' allows for a certain tolerance, in the sense that small variations of pitch are not perceived as *melodically different.* Perhaps some of the finest gradations may be heard in the *srutis* of South Indian melody. In my own experience, intervals smaller than 1/6 of a tone (approximately 35¢) begin to take on the character of enharmonic shadings of pitch rather than functioning as distinct tones. Traditionally, theory has referred to such microtonal variations as *commas* and *schismas*.

On the other hand, the relations 'higher' and 'lower' tend to be divided *musically* into two general types of melodic interval: those which fall within one critical bandwidth (roughly an 8/7 ratio of frequencies) are said to move by *step,* and are referred to as *tones* and fractions thereof (e.g. semitone, quartertone, sixthtone); larger intervals are often described as *leaps*.

Claudius Ptolemy describes the situation more precisely by identifying three classes of non-unison intervals: homophones, concords, and the melodic. Homophones are manifestations of the *octave-equivalence phenomenon*. Concords are those intervals 'nearest' to the homophones, that is, the ratios 3/2 (diapente) and 4/3 (diatessaron) and their octave-composites. Melodic intervals include the difference between the primary concords (the tone 9/8) and other 'nearest' epimoric ratios of the form

$$(n+1):n \tag{2}$$

Generalizing from this approach, we begin by examining the relations between *simultaneously* (rather than *successively)* sounded pitches. The advantage of such a method is that very slight variations of intonation, which might not disturb our perception of intervals when pitches are sounded one after the other, may be clearly distinguished when two tones are heard at the same time.

One may proceed as follows: first, by defining an extended set of concords – intervals *which may be accurately determined by perception alone;* second, by considering the set of all possible melodic movements between these intervals. This would be a repertoire of melodic steps and leaps, a complete range of *possible musical intervals,* each of which could be accurately reproduced in reference to a virtual implied third pitch.

Once again, perception suggests a rough division into three classes. In detuning one sound from unison with another, there is a region of pleasant beating, followed by a region of *roughness.* Throughout this entire range it is often difficult to hear two distinct pitches. Instead, there is a tendency to perceive a single average pitch and amplitude modulation as expressed by the well-known trigonometric identity

$$\cos u + \cos v = 2\cos((u+v)/2)\cos((u-v)/2) \tag{3}$$

Once the distance between pitches reaches an 8/7 interval or more, the sensation of mutually generated roughness tends to be replaced by a smoother coexistence of the two separate pitches, and it becomes considerably easier to distinguish the *sonorous properties* of intervals. These variations of sound quality, which do not primarily have to do with the melodic notions 'higher' and 'lower,' may be called *harmonic* relationships. The phenomenon of octave equivalence, which allows men and women (or boys) singing at different octaves to believe that they are producing 'the same' pitch, is one clear example.

Harmonic relationships between two sounds are largely determined by beating and unison relationships between partials, as well as characteristic patterns produced between relative beating rates on different critical band regions, which are associated (in well-tuned sounds) with a sensation of *spectral fusion.* The phenomena described above in reference to the detuning of a unison are now replicated between various spectral components. Partials lock into periodic patterns, creating auditory phenomena similar to the visual patterns made by a stroboscope.

This process may be effectively investigated by constructing just intervals using additive synthesis, eliminating common partials, and hearing at what point the distinctive qualities defining specific intervals begin to disappear. To a lesser degree, harmonic relationships may also be deduced from the periodicity of a composite waveform. However, listening for beating in a non-unison interval produced between two pure sinewaves and attempting to tune it exactly is considerably more difficult than the same task with two spectrally rich tones.

Thus, in the case of simultaneously sounded pitches, discrete *steps* may most readily be found by *harmonically* examining the larger intervals, which *melodically* would be considered to be *leaps.* Those intervals that may be *precisely and repeatedly produced by perception alone* I refer to as *tuneable intervals.*

III. Tuneable Intervals

The notion of a range of tolerance within which an interval may be considered to be *de*-tuned or *mis*-tuned clearly implies that infinitely many *other* rational relationships situated in the range are not heard as distinct intervals at all. Slight detuning, producing slow *Leslie-speaker* phasing or beating in the form of *vibrato,* is often a very beautiful musical inflection of an interval that in no way damages its comprehensibility. A highly developed culture of such inflections applied within a very precisely tuned context is evident, for example, in the *gamakas* of South Indian classical music. Alternately, harmonically conceived keyboard music by J.S. Bach or Frederic Chopin, for example, is based on the ranges of tolerance inherent in the theoretically "out of tune" systems of well-temperaments or equal temperament.

At the same time, there are often *several* variations of fine-tuning possible within a given tempered interval-class, each of which may be well determined by listening for and eliminating beats. One common example is the tritone 10/7 (617.5¢) and the diminished fifth 7/5 (582.5¢), which differ by a melodic sixth-tone 50/49 (35¢). The tempered interval 600¢ found on MIDI keyboards lies in-between the two just ratios.

In addition to such variations of the well-known intervals, there are also many intervals which lie outside the range of tolerance of the 12 tempered pitch-classes but which may nevertheless be precisely heard. Most familiar in the context of Arabic music: the 'neutral' third of Zalzal (27/22 or 354.5¢).

In general, there are more intervals that can be tuned by ear than the tempered system distinguishes, but this collection of intervals is also somehow bounded. This suggests that a provisional set of *tuneable intervals* may be found. Some criteria suggest themselves for the scope of such an investigation:

(1.) **Timbre:** Each interval may be tested with both electronic additive synthesis timbres, and with acoustic instrumental sounds. When testing with electronic sounds it is useful to eliminate the clearly audible beating between common partials. In acoustic instrument sounds, the continuous variation of amplitude in spectral components reduces the effectiveness of this kind of beating as a primary tuning cue.

(2.) **Range for each interval:** It is useful to determine the range within which each interval may be tuned. It is possible that a quantitative relationship might be deduced between the individual ranges of tuneability and the register of primary difference tones and especially of the *periodicity pitch* (a virtual fundamental common to both pitches).

(3.) **Harmonic Distance:** Since intervals become increasingly difficult to hear as their ratio-numbers increase, it might be assumed that above a certain HD intervals are no longer directly tuneable, and would instead have to be constructed from simpler intervals.

(4.) **Octave Equivalence:** Some intervals may quite readily be tuned in a wide voicing (larger than an octave) whilst remaining difficult or impossible to tune in closed position. Once a set of tuneable intervals has been established, the effect of octave equivalence within this gamut may be investigated, particularly in relation to the possible formations of musical scales and modes.

(5.) **Range of Tuneability:** Due to critical band phenomena, small intervals are eventually perceived more readily as an amplitude-modulated average value than as two distinct pitches, as discussed above. Thus, there is a smallest non-unison tuneable interval. Similarly, as the intervals become larger, after a certain point the ability to perceive and differentiate harmonic phenomena becomes negligible. It is possible that the formula for harmonic distance must be appropriately modified or bounded to take these extreme conditions into better account.

(6.) **Distribution of Intervals:** In some cases, several similarly complex intervals happen to fall very close to each other, causing their harmonic qualities to be less readily distinguished (e.g. 13/9 and 16/11 are separated by 144/143, or about 12¢). Therefore, the relationship between tuneability and the distribution of ratios also merits further examination.

An ongoing empirical investigation (using acoustic and electronic sounds in various registers) has to date identified 122 23-limit tuneable intervals in the range 1/1 (0¢) to 28/1 (5769¢). The complete list of all distinct lowest-terms ratios between the first 28 partials, together with information about prime limit, melodic distance, harmonic distance, and an empiric evaluation of tuneability on a four-level scale (impossible, very difficult, average, easy; numerically represented as 0–3) may be found as an Appendix below.

These harmonic relationships have been used to construct the provisional form of the tuning algorithm described in this paper.

IV. Tuneable Melodic Steps

When the 122 tuneable intervals are used to define a set of pitches both above and below a given starting note, and the doubled occurrence of the unison 1/1 is eliminated, 243 distinct pitches remain. By generating a 2-dimensional 243x243 array, all of the intervals spanning these pitches may be determined. Once duplicates have been filtered, this produces a list of 3997 unique *tuneable melodic intervals* ranging from a unison (1/1 or 0¢) to just under 10 octaves (784/1 or 11537.7¢). Each of these melodic intervals implies a *virtual third pitch,* which (if sustained while successively sounding the tones forming the melodic interval) will form tuneable intervals to both initial pitches.

The fine-tuning algorithm described below uses this array of intervals as a lookup table, based on the premise that it encompasses tuneable intervals "one-step-removed" and may therefore effectively model how our brains could deduce possible connections between pitches.

A few statistical observations about this list are tabulated to indicate how finely it resolves the glissando-continuum:

TABLE I.
Tuneable Melodic Intervals by Octave

Octave	# of Intervals	Mean Step	Smallest Step	Largest Step
all	3997	2.89¢	0.07¢	62.96¢
1	785	1.52¢	0.07¢	8.34¢
2	720	1.67¢	0.07¢	8.34¢
3	667	1.80¢	0.07¢	8.34¢
4	600	2.00¢	0.07¢	8.34¢
5	431	2.78¢	0.14¢	11.35¢
6	310	3.87¢	0.36¢	13.65¢
7	225	5.29¢	0.40¢	23.34¢
8	133	9.09¢	0.79¢	28.27¢
9+	126	15.38¢	2.38¢	62.96¢

In future development, I plan to revise this array as follows: rather than building it from an empirically determined tuneable interval set, I would prefer to find a quantitative HD value above which no more tuneable intervals may be found. Then, all possible ratios found to lie within this limiting value might further be constrained by an overall range of tuneability (see discussion above), and by ranges found for each individual interval.

Based on the particular experience and intervallic discrimination of a given listener, it would then be possible to dynamically adjust the array by specifying a limiting HD and several range-related parameters.

V. Triads

So far, the discussion has been limited to the case of intonation in dyads. Clearly the situation increases in complexity with each additional note comprising an aggregate of pitches. At the same time, the most important qualitative (musical) differences are represented in the three simplest cases: *melodic intervals* (two successive pitches); *harmonic intervals* (two simultaneous pitches); *chords* (three or more simultaneous pitches).

At this point, I limit my discussion of chords to the case of triads. In the future development of the algorithm proposed here, I anticipate that it will be possible to generalize to structures involving more pitches. Eventually, such study might also be connected to the more statistical properties of large microtonal *aggregates* explored by Iannis Xenakis and by some of the spectralist composers (Gérard Grisey, Tristan Murail, Oliver Schneller). It will also require work to understand how larger aggregates form modal sets, which become clearly defined in memory even when not all pitches are sounding.

In analyzing the problem of fine-tuning an arbitrarily formed microtonal triad, it is useful to consider that there are not only three different pitches, but also *three intervals* formed between *pairs* of pitches. Taking the three pitches in ascending order (expressed as frequencies)

$$F1 < F2 < F3 \quad (4)$$

the three intervals may be summarized mathematically as

$$(F3/F1) = (F3/F2)(F2/F1) \quad (5)$$

Looking at the triad expressed in this way, and comparing the three dyads to the previously discussed set of tuneable intervals, four possibilities emerge. *In any triad, either none, one, two, or all three of the component dyads form tuneable intervals.* In the 'two' case one might note two sub-classes, depending on whether the *outer* interval is tuneable or not.

Since the composite sound of a triad is, in some way, a superimposition of three interference patterns between pitch-pairs, it seems intuitively correct to postulate that triads made up from two or three tuneable dyads will generally sound more stable and comprehensible than those with none or only one. Thus, assuming that the purpose of fine-tuning is to maximize clarity and variety within a large range of possible sound structures, any algorithm that *favors the possibility of two or more tuneable interval relationships* will produce results with a pronounced acoustical advantage.

Each triad made up from the same dyads has two possible symmetric forms, which (following Partch) I distinguish as *otonal* or *utonal.*

In the case of dyads, this may be imagined as the difference between taking an interval upward or downward from a given starting pitch. The resulting sound structure is related by a transposition of register, but otherwise exactly identical.

In the case of three pitches, written as above, the two smaller intervals F2/F1 and F3/F2, taken together, "add up" to form the outer interval F3/F1. It is possible to reverse the order of these two smaller intervals by defining a new pitch

$$F2^* = (F3/F2) \cdot F1 \quad (6)$$

(So F2*/F1 = F3/F2 and F3/F2* = F2/F1.)

If the two chord structures F1:F2:F3 and F1:F2*:F3 are expressed in lowest terms, either they will both turn out to be identical (if F2* = F2, i.e. F3/F2 = F2/F1) or one of these two forms may be found to consist of "smaller" numbers. In this case, define "smaller" in the following sense: if the outer terms of both forms are the same, take the triad with a smaller middle term; if not, take the one with a smaller first term. This I would then call the *otonal* form.

To briefly explain this idea, take as an example the relationship between a major and minor triad. Both are composed of a major third (5/4) and a minor third (6/5) adding up to a perfect fifth (3/2). As chords, they may be represented by the frequency ratios 4:5:6 (major) and 10:12:15 (minor). The numbers indicate that the minor triad occurs later in a harmonic (overtone) series than the major triad, which is therefore acoustically simpler (easier for the ear to analyze) and which, according to the definition above, is *otonal.*

Consider the property that any tuned aggregate of pitches shares not only a *common fundamental frequency* but also a *least common partial.* It is thus also possible to express chord proportions in relation to their common partial. Still thinking of overtone series, in the case of both 4:5:6 and 10:12:15 the least common partial will be 60 (the least common multiple in both cases). Thus, *in relation to this common partial* (which, in a well-tuned triad, may be acoustically perceived as part of the composite sound) the minor triad may be expressed as 1/6:1/5:1/4 and the major as 1/15:1/12:1/10.

So, considered downward *(utonally),* the minor triad takes smaller numbers. Namely, when building a subharmonic (undertone) series downward from the common partial, the minor triad will occur sooner than the major, and so I refer to it as *utonal.*

The symmetry inherent in this argument has been compelling to many music theorists, including Rameau, Riemann and Partch, particularly as a way of "explaining" the minor triad (in spite of its dissonant difference tones) and also as a way of generalizing major-minor tonality. However, the perception of chord "stability" (which tonality requires) is based on a psychoacoustic sensation of fusion produced by harmonic spectra. Utonal sounds are thus by definition less stable than otonal sounds (because they are further away from their fundamental periodicity pitch). As utonal structures become increasingly complex, their conceptual symmetry to otonal counterparts is no longer acoustically perceptible.

Nevertheless, it is certainly possible and musically fruitful to investigate a list of simpler triads in which this quality *is* maintained. One interesting such example is the septimal triad 6:7:8 and its utonal counterpart 21:24:28.

VI. Development of the Algorithm

The problem might be summarized as follows: given two frequencies extracted from a spectral aggregate (e.g. a woodwind multiphonic timbre), what might be an effective method for extrapolating to harmonically interesting third pitches?

First, there is the problem of deducing potentially *perceptible* harmonic relationships (ratios) between the two extracted pitches. Then, once one of these ratios has been selected, the choice of a third pitch is evaluated based on the intervals it forms with both initial pitches, as well as by the overall sonorous qualities of the triad all three generate.

Imagine the three pitches forming a triangle. By slightly adjusting each vertex it is possible to find a proportionally ideal structure. The first vertex need only be adjusted if there is an external tuning standard in place (for example, A-440). The second vertex is adjusted to the first, producing a tuneable melodic step, which may or may not itself be a tuneable interval. Then any number of tuned possibilities exist for the third vertex, at least some of which produce two tuneable intervals to both initial pitches.

This model led to the current harmonic microtuning algorithm, which exists in the form of an external object, programmed in C, and compiled to run in the MaxMSP environment. This external has been implemented within a Max patch, which allows for the fine-tuning of up to three pitches, in relation to each other and to a reference frequency, and according to three additional user-specified parameters: tolerance, pitch-class scaling, and prime limit.

The program uses the list of 3997 tuneable melodic intervals as a lookup-table, searching to find the nearest possible match within a desired prime limit. If this nearest result falls outside of a user-specified tolerance range, then it is immediately output. Otherwise the program searches for the *simplest* result within the desired range, evaluated by minimizing a harmonic-distance sum.

This sum is weighted by the choice of a pitch-class scaling value, which serves to either favor the simplicity of the sounding interval (value 0), the spelling of the microtonal pitch within a notated system (value -1), or the pitch-class relationship to its reference pitch (value 1). Intermediate values produce a linear interpolation between the evaluated harmonic distances. Pitch-class HDs are computed by ignoring the octave prime dimension 2 (factoring out all the 2's in the ratios). The resulting harmonic space is called *projection space.*

Input to the external is distributed between ten available 'inlets.' Following the right-to-left logic of Max, inlet 10 accepts integers and determines a prime limit ranging from Pythagorean intonation (intervals generated by the numbers 2 and 3 and their powers) to the limits of tuneability (intervals made from the primes 3, 5, 7, 11, 13, 17, 19, 23). In future implementations, it would be interesting to have an on-off choice for each prime, allowing, for example, a set of intervals generated by the primes 3 and 7 only.

Inlet 9 accepts pitch-class scaling, in the form of a float value between -1 and 1. If 0 is entered, the algorithm evaluates intervals as they sound, without considering pitch-class. Thus, if a G, a minor seventh above A, is input as the first frequency to be tuned, and the tolerance range and prime limit allow it, the algorithm will prefer 9/5 (raising the G by a comma) to 16/9 (Pythagorean G). However, if the same pitch class is entered two octaves lower, as a G that is a major ninth below A, then the algorithm would prefer 4/9 (Pythagorean G). In both cases, the simplest *sonority* is chosen.

Sometimes, however, it might be preferable for the algorithm to choose in the manner of traditional modes and scales, repeating identically in each octave. (In this case it should retune *consistently* whenever it receives a G.) The decision may be weighted in favor of pitch class relationships to the reference frequency (values between 0 and 1), or in favor of the microtonal spellings using the Extended Helmholtz-Ellis JI Pitch Notation (values between 0 and -1). This second choice might produce slightly more complex intervals but has the advantage of keeping the notated pitches simpler, useful when dealing with a written score or whilst algorithmically generating pitches to be written down and played by other instruments.

Inlet 7 is the reference frequency (F0), expressed as a float value in Hz. I generally leave this at the standard tuning reference (440 Hz), but it is possible to use any frequency (for example, any tempered pitch in any octave).

The first six (of ten total) inlets are taken up (pairwise) with the three frequencies (F3, F2, F1), which may each be input in one of three possible numerical representations. In the following description, the terms 'inlet 1' and 'inlet 2' are used an as example: inlets 3&4 and 5&6 may be imagined as behaving identically.

If an **exact just intonation ratio** to F0 is desired, namely a pitch, *which is already tuned and ought to be left alone,* then inlet 1 and inlet 2 must both receive nonzero positive integers. If an **absolute frequency value** is to be interpreted, inlet 2 should receive a 0 and inlet 1 a positive float value.

If the input desired is **MIDI+cents** (which offers a greater degree of precision) then inlet 1 must receive a negative value, which may be calculated from the desired MIDI note. (The MIDI value of the note is multiplied by -1 and added to -1000, an arbitrary offset value, which must also be adjusted by the distance of the tempered reference frequency from A4 MIDI value 69.) In this case inlet 2 accepts positive or negative float values for deviation from the MIDI value in cents.

Each incoming frequency is tuned in relation to a *reference:* F1 is tuned to F0, F2 is tuned to F1, and F3 is tuned to either F2 or F1, whichever possibility offers the best overall result. The ability to input a ratio allows the user to specify complex just intonation intervals, which will not be retuned, as components of the structure.

VII. Current Implementation (Micromælodeon I)

The current algorithm is limited to making decisions about the fine-tuning of one, two or three pitches, received successively, sounding simultaneously. It has been implemented as the core of a virtual instrument called Micromælodeon I. Up to three sounds may be generated and selectively fine-tuned, using a simple wave-shape-morphing synthesis method combined with a filtergraph used to simulate resonances and formants.

It is possible that the three-pitch 'triangulation' process described in this paper may be effectively generalized to tune structures of up to twelve pitches related in a network as follows:

(1.) F0, F1, F2, F3 function as described above.

(2.) F4 is tuned to F0, F1, and F3.

(3.) F5 is tuned to F0, F2, and F3.

(4.) F4 and F5, combined respectively with each of F1, F2, F3 (with reference F0) produce the next six pitches – F6 through F11.

(5.) F4 and F5 (with reference F0) produce F12.

A second instrument to investigate this extended model is being planned and programmed, to be followed by empirical testing of the algorithm's performance with trained musicians, the design of hardware control interfaces for virtual instruments and implementation of a learning memory which would facilitate developing harmonic decision-making over larger musical time-scales (phrase, section, entire piece).

I anticipate that developing this 'memory' will continue my previous work with 'crystal growth' algorithms in harmonic space, which enable stochastic generation of harmonically compact pitch-clusters.

VIII. Conclusion

In informal testing, given appropriately chosen reference pitches and parameters, the Micromælodeon is readily able to find classical sets of pitches: among others, the major and minor scales tuned in Just Intonation (Ptolemaic tense [syntonon] diatonic); the 7-limit srutis used in Indian music. At the same time, it suggests subtle fine-tunings of more ‘dissonant’ equal-tempered chords, based on relationships of higher partials, revealing complex tonal relationships underlying ‘atonal’ sounds.

It is hoped that the algorithm presented here will provide a foundation for new electronic instruments, which allow for precise *musical* investigations of the phenomena of harmonic perception, by implementing well-formed *descriptive* (rather than pro- or pre-scriptive) principles of those relations between pitches which do not have to do exclusively with “higher” and “lower” (i.e. *harmony*).

APPENDIX: TABLE OF TUNEABLE INTERVALS

The intervals below are taken from the first 28 partials, ordered by rising HD-product (numerator multiplied by denominator). The complete list of unique lowest terms ratios was truncated at the point after which no more tuneable intervals were found. Boldface font indicates extensions of the original three-octave tuneable set. Italic font indicates intervals that were not found to be tuneable. The informal ‘degree’ column shows, on a scale from 0 to 3, my own assessment of how difficult the tuning task was using acoustic stringed instruments (violin, viola, violoncello, contrabass).

TABLE II.
TUNEABILITY OF INTERVALS SORTED BY HD-PRODUCT

Ratio (num)	Ratio (den)	Degree (0–3)	HD-product	Prime Limit	Size (cents)
1	1	3	1	1	0
2	1	3	2	2	1200
3	1	3	3	3	1901.955001
4	1	3	4	2	2400
5	1	3	5	5	2786.313714
3	2	3	6	3	701.9550009
6	1	3	6	3	3101.955001
7	1	3	7	7	3368.825906
8	1	3	8	2	3600
9	**1**	**3**	9	**3**	**3803.910002**
5	2	3	10	5	1586.313714
10	**1**	**3**	10	**5**	**3986.313714**
11	**1**	**3**	11	**11**	**4151.317942**
4	3	3	12	3	498.0449991
12	**1**	**3**	12	**3**	**4301.955001**
13	**1**	**3**	13	**13**	**4440.527662**
7	2	3	14	7	2168.825906
14	**1**	**2**	14	**7**	**4568.825906**
5	3	3	15	5	884.358713
15	**1**	**2**	15	**5**	**4688.268715**
16	**1**	**2**	16	**2**	**4800**
17	**1**	**1**	17	**17**	**4904.95541**
9	2	3	18	3	2603.910002
18	**1**	**1**	18	**3**	**5003.910002**
19	**1**	**1**	19	**19**	**5097.513016**
5	4	3	20	5	386.3137139
20	**1**	**1**	20	**5**	**5186.313714**
7	3	3	21	7	1466.870906
21	**1**	**1**	21	**7**	**5270.780907**
11	2	3	22	11	2951.317942
22	**1**	**1**	22	**11**	**5351.317942**
23	**1**	**1**	23	**23**	**5428.274347**
8	3	3	24	3	1698.044999
24	**1**	**1**	24	**3**	**5501.955001**
25	**1**	**1**	25	**5**	**5572.627428**
13	2	3	26	13	3240.527662
26	**1**	**1**	26	**13**	**5640.527662**
27	**1**	**1**	27	**3**	**5705.865003**
7	4	3	28	7	968.8259065
28	**1**	**1**	28	**7**	**5768.825906**
6	5	3	30	5	315.641287
10	3	3	30	5	2084.358713
15	2	3	30	5	3488.268715
11	3	3	33	11	2249.362941
17	**2**	**2**	34	**17**	**3704.95541**
7	5	3	35	7	582.5121926
9	4	3	36	3	1403.910002
19	**2**	**2**	38	**19**	**3897.513016**
13	3	3	39	13	2538.572661
8	5	3	40	5	813.6862861
7	6	3	42	7	266.8709056
14	3	3	42	7	2666.870906
21	**2**	**2**	42	**7**	**4070.780907**
11	4	3	44	11	1751.317942
9	5	3	45	5	1017.596288
23	**2**	**2**	46	**23**	**4228.274347**
16	3	3	48	3	2898.044999
25	**2**	**2**	50	**5**	**4372.627428**
17	3	3	51	17	3003.000409
13	4	3	52	13	2040.527662
27	**2**	**1**	54	**3**	**4505.865003**
11	5	2	55	11	1365.004228
8	7	2	56	7	231.1740935
19	3	3	57	19	3195.558015
12	5	3	60	5	1515.641287
15	4	3	60	5	2288.268715
20	3	3	60	5	3284.358713
9	7	3	63	7	435.0840953
13	5	2	65	13	1654.213948
11	6	2	66	11	1049.362941
22	3	3	66	11	3449.362941
17	4	2	68	17	2504.95541
23	3	3	69	23	3526.319346
10	7	2	70	7	617.4878074
14	5	3	70	7	1782.512193
9	**8**	**1**	**72**	**3**	**203.9100017**
25	**3**	**1**	75	**5**	**3670.672427**
19	4	2	76	19	2697.513016
11	7	2	77	11	782.4920359
13	6	2	78	13	1338.572661
26	**3**	**2**	78	**13**	**3738.572661**
16	5	3	80	5	2013.686286
12	7	2	84	7	933.1290944
21	4	2	84	7	2870.780907
28	**3**	**1**	84	**7**	**3866.870906**
17	5	2	85	17	2118.641696
11	8	1	88	11	551.3179424
10	*9*	*0*	*90*	*5*	*182.4037121*
18	5	3	90	5	2217.596288
13	7	2	91	13	1071.701755
23	4	2	92	23	3028.274347
19	5	2	95	19	2311.199302
11	9	1	99	11	347.4079406
25	4	2	100	5	3172.627428
17	6	2	102	17	1803.000409
13	8	2	104	13	840.5276618
15	*7*	*0*	*105*	*7*	*1319.442808*
21	5	1	105	7	2484.467193
27	4	1	108	3	3305.865003
11	*10*	*0*	*110*	*11*	*165.0042285*
22	5	1	110	11	2565.004228

16	7	0	112	7	1431.174094
19	6	1	114	19	1995.558015
23	5	2	115	23	2641.960633
13	9	1	117	13	636.61766
17	7	1	119	17	1536.129503
15	8	1	120	5	1088.268715
24	5	2	120	5	2715.641287
14	9	1	126	7	764.9159047
18	7	2	126	7	1635.084095
13	10	2	130	13	454.2139479
26	5	1	130	13	2854.213948
12	11	0	132	11	150.6370585
19	7	0	133	19	1728.68711
27	5	0	135	5	2919.551289
17	8	0	136	17	1304.95541
23	6	1	138	23	2326.319346
20	7	2	140	7	1817.487807
28	5	2	140	7	2982.512193
13	11	0	143	13	289.2097194
16	9	0	144	3	996.0899983
25	6	1	150	5	2470.672427
19	8	1	152	19	1497.513016
17	9	0	153	17	1101.045408
14	11	0	154	11	417.5079641
22	7	1	154	11	1982.492036
13	12	0	156	13	138.5726609
23	7	0	161	23	2059.448441
15	11	0	165	11	536.9507724
21	8	0	168	7	1670.780907
24	7	1	168	7	2133.129094
17	10	0	170	17	918.6416956
19	9	0	171	19	1293.603014
25	7	0	175	7	2203.801521
16	11	1	176	11	648.6820576
20	9	0	180	5	1382.403712
14	13	0	182	13	128.2982447
26	7	0	182	13	2271.701755
23	8	2	184	23	1828.274347
17	11	0	187	17	753.6374671
27	7	1	189	7	2337.039096
19	10	0	190	19	1111.199302
15	13	0	195	13	247.741053
18	11	0	198	11	852.5920594
22	9	0	198	11	1547.407941
25	8	1	200	5	1972.627428
17	12	0	204	17	603.0004086
23	9	0	207	23	1624.364346
16	13	0	208	13	359.4723382
19	11	0	209	19	946.1950738
15	14	0	210	7	119.4428083
21	10	0	210	7	1284.467193
27	8	1	216	3	2105.865003
20	11	0	220	11	1034.995772
17	13	0	221	17	464.4277477
25	9	0	225	5	1768.717426
19	12	0	228	19	795.5580153
23	10	0	230	23	1441.960633
21	11	0	231	11	1119.462965
18	13	0	234	13	563.38234
26	9	0	234	13	1836.61766
17	14	0	238	17	336.129503
16	15	0	240	5	111.7312853
19	13	0	247	19	656.9853544
28	9	1	252	7	1964.915905
23	11	0	253	23	1276.956405
17	15	0	255	17	216.6866948
20	13	0	260	13	745.7860521
24	11	0	264	11	1350.637059
19	14	0	266	19	528.6871097
27	10	0	270	5	1719.551289
17	16	0	272	17	104.9554095
21	13	0	273	13	830.2532456
25	11	0	275	11	1421.309485
23	12	1	276	23	1126.319346

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