Modulus p Rhythmic Tiling Canons and some implementations in Open Music visual programming language

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ABSTRACT

The concept of rhythmic canons, as it has been introduced by mathematician Dan Vuza in the 1990s, is the art of filling the time axis with some finite rhythmic patterns and their translations, without onsets superposition. The musical notion have been linked with some mathematical results, and since then, its mathematical study has always followed a will of picturing every new results in the visual programming language OpenMusic, which enables mathematicians and composers to work together. In this paper we present some new results in an enriched version of rhythmic tiling canons, where some controlled superposition are allowed. This enhanced version of rhythmic tiling canons is presented at the beginning of this article, as well as main constructive results, because it is fairly recent. Then the paper focuses on the presentation of some generative transformations, building canons with the same superposition. The latter is at the heart of the study of canons allowing superposition, because they are the key of linking them back to seminal canons. In order to help composers experiment with these new canons, every constructive method has been implemented in OpenMusic as part of the MathTools environment.

1. INTRODUCTION

1.1 History

Since the publication of the four-parts paper by Dan Tudor Vuza establishing the theoretical foundations of the construction of Rhythmic Tiling Canons [1], many efforts have been made in the integration of these algebraic tools within some environments for computer-aided music analysis and composition. There are many reasons for studying the computational aspects of this music-theoretical model, starting from the necessity of knowing exactly the number of possible rhythmic patterns which tile the time axis by translation, in such a way that no inner periodicity is found neither in the generating pattern (also called *inner rhythm*) nor in the pattern providing the entries of the various voices of the canon (which is sometimes called the *outer rhythm*). The first results in the computation and classification of Tiling Canons without inner periodicity,

commonly known as *Vuza Canons*, have been obtained once the theoretical model has been originally integrated in OpenMusic visual programming language [2] and this has provided composers with a panoply of new tools and opened interesting questions about more general models of tiling process (see [3] for a survey on the existing OpenMusic objects that have been recently integrated in the MathTools environment, together with other algebraic-oriented models for computer-aided composition and analysis). We will not present the models in detail, since they have been presented many times in previous conferences (see, in particular [4], [5], [6], [7]) and they are therefore relatively familiar to the computational musicology community of ICMC and SMC. What all the previous computational models have in common is to focus on tiling constructions in which each time-point is occupied by one and only one attack of a voice of the canon. Although this way of tiling the time-axis has been largely used by composers (see, e.g., the two volumes of the OM Composers Book [8] and [9]), the model is intrinsically monophonic, in contrast with the polyphonic character of the traditional canonic process. This attempt at taking into account polyphony within a general model of Tiling Canons not only asks for new algorithms but, first of all, requires a change of perspective in the theoretical foundation of the model, as we will see in the next section.

1.2 Why modulus p?

Tiling by translation is a pretty hard task, so why would we want to add the constraint of managing the overlaps? The idea actually came from Emmanuel Amiot [10] while working with polynomial notation. Rhythmic tiling canons are equivalent to products of two polynomials with coefficients in $\{0, 1\}$ (for more details about the relevancy of the polynomial notation, the reader can refer to [6]). But since the property of being a polynomial with coefficients in $\{0, 1\}$ is not closed under product, the idea of working in the polynomial field $\mathbb{F}_2[X]$ came to mind, and by extension, the complete concept of modulus p canons.

It appeared that this idea involved huge improvements of the notion, both mathematically and musically.

Modulus p tiling enriches classical rhythmic tiling canons with harmony, allowing notes superposition. Thus, it breaks the monotony of a strict tiling with some controlled covering. It is also a powerful mathematical tool, because thanks to some of its properties, it is extremely easy to compute a modulus p tiling by translating any given finite pattern.

For the sake of clarity, we will omit the proof of the math-

Figure 1. The Abadja rhythm.

ematical statements in this paper, although proved else-where.

2. DEFINITIONS AND EXAMPLES

Musically, a rhythmic tiling canon (RTC), is a canon where only the onsets matter. The pitch and length of the notes are not relevant, and we impose the pattern of onsets and its translates to tile the time axis represented by the discretized line $\mathbb{Z}$. That is to say that a finite rhythmic pattern (the *inner rhythm*) is played at some beats (the *outer rhythm*) to obtain one and exactly one note at every pulsation. Such a canon is equivalent to a tiling of the line $\mathbb{Z}$ by a finite integers set and its translates, i.e. to a direct sum of two integers sets being equal to $\mathbb{Z}$. Lagarias and Wang [11, theorem 5] have proved that RTC can be reduced to a direct sum of two *finite* integers sets being equal to $\mathbb{Z}_N$ for some $N \in \mathbb{N}^*$.

For more details on the subject of RTC, the interested reader is invited to refer to [4] in 2002 ICMC proceedings.

Thanks to Vuza's modelisation of rhythm [12], we will denote a rhythmic pattern as a set of integers containing 0.

Example 1: The Abadja rhythm, from traditional music of Ghana, in figure 1, can be represented by the set $\{0, 2, 4, 5, 7, 9, 11\}$.

A modulus p rhythmic tiling canon (RT_pC) is like a classical RTC, except that one wants to obtain one and exactly one note modulus p at every pulsation.

Definition 1: The couple of rhythmic patterns (A, B) is a modulus p rhythmic tiling canon of $\mathbb{Z}_N$ iff $C = A + B$ mod N the set sum modulus N verifies for all n in $\mathbb{Z}_N$ $\mathbb{1}_C(n) = 1$ mod p, where $\mathbb{1}_C$ is the multiset characteristic function.

Example 2:
The couple $(A, B) = (\{0, 1, 3, 6\}, \{0, 1, 2, 3, 4, 5, 6, 7\})$ verifies

$$
\begin{aligned}
C &= \{0, 1, 3, 6\} + \{0, 1, 2, 3, 4, 5, 6, 7\} \\
&= \{0, 1, 1, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 6, \\
&\quad 7, 7, 7, 7, 8, 8, 8, 9, 9, 10, 10, 11, 12, 13\} \\
&= \{0, 0, 0, 0, 1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 4, 4, \\
&\quad 4, 4, 5, 5, 5, 5, 6, 6, 6, 6, 7, 7, 7, 7\} \bmod 8
\end{aligned}
$$

And $\mathbb{1}_C(n) = 4 = 1$ mod 3 for all $n \in \mathbb{Z}_8$.

Hence (A, B) is a RT_3C of $\mathbb{Z}_8$.

Definition 2: A RT_pC of $\mathbb{Z}_N$ is *compact* if the sum $A + B$ modulus N is equal to the sum without projection in $\mathbb{Z}_N$.

Example 3:
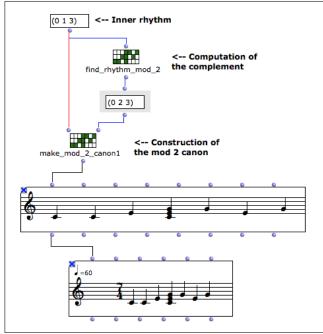$(\{0, 1, 3\}, \{0, 2, 3\})$ is a compact RT_2C of $\mathbb{Z}_7$ (see Fig. 2).

Figure 2. An OpenMusic patch, finding the outer rhythm forming a compact RT_2C, and representing it in traditional music notation.

3. HOW TO TILE MODULUS P?

A theorem using Galois Theory and polynomial notation firstly noticed by Amiot in [13] entails that every rhythmic pattern tiles modulus p, and even better, that every pattern gives a *compact* RT_2C.

Those properties allow us to construct a greedy algorithm (Algorithm 1) which returns the outer rhythm B and the period N if given an inner rhythm A such that (A, B) is a compact RT_2C of $\mathbb{Z}_N$.

Algorithm 1: GreedyTiling
Require: $A \subset \mathbb{N}$ finite and $0 \in A$
1: $B = \{0\}$, $N = \max A$
2: **while** $(A + B) \neq \{1, \ldots, 1\}$ mod 2 **do**
3:     $i \leftarrow$ first index such that $(A + B)[i] \neq 1$ mod 2
4:     **if** $i \neq (N + 1)$ **then**
5:         $B := B \cup \{i\}$
6:         $N := i + \max A$
7:     **else**
8:         break while
9:     **end if**
10: **end while**
11: **return** (B, N)

In figure 2, one can see this very fast algorithm implemented in Open Music.

4. VUZA RT_PC

For the classical RTC study, mathematicians and composers spearhead at Vuza canon. Defined firstly as being non-periodic, they can be described in an easier way, that one can adapt modulus p.

Definition 3: If (A, B) is a RTC of $\mathbb{Z}_N$, then for all $k \in$
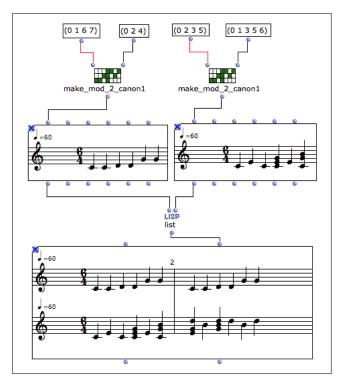
Figure 3. An OpenMusic patch representing two RT_2C of $\mathbb{Z}_{12}$. The upper one is not Vuza whilst the second one is.

$\mathbb{N}^*$, the k-concatenation $\overline{A}^k$ of A is defined as

$$\overline{A}^k = A + \{0, N, 2N, \ldots, (k-1)N\}.$$

The tiling property is closed under concatenation transformation as proved by Amiot in [13]:

Property 1: (A, B) is a RTC of $\mathbb{Z}_N$ iff $(\overline{A}^k, B)$ is a RTC of $\mathbb{Z}_{kN}$.

The following property also proved in the same paper [13] gives us a characterization of Vuza canons one can interpret for modulus p canons:

Property 2: Every RTC can be deduced by concatenation (and duality) transformations from Vuza canons and the trivial canon $(\{0\}, \{0\})$.

This means that Vuza canons can also be defined as minimal under the concatenation transformation.

Definition 4: A RT_pC (A, B) is a Vuza RT_pC if both A and B are not derived from the concatenation of smaller rhythmic patterns.

This definition allows to prove the following theorem ([14]) giving a NSC to know wether a RT_2C is a Vuza canon.

Theorem 1: The RT_2C (A, B) is a Vuza RT_2C iff given A the Algorithm 1 of figure 2 returns B and if given B it returns A.

Example 4: $(\{0, 2, 3, 5\}, \{0, 1, 3, 5, 6\})$ is a compact Vuza RT_2C of $\mathbb{Z}_{12}$ whereas $(\{0, 1, 6, 7\}, \{0, 2, 4\})$ is a compact RT_2C of $\mathbb{Z}_{12}$ that is not Vuza. The reader can immediately see in figure 3 picturing both these canons that the first one is derived from the concatenation of a smaller one: $(\{0, 1\}, \{0, 2, 4\})$, depicted before the first bar.

Thanks to this theorem, we can verify very easily if a rhythmic pattern A will produce a Vuza compact RT_2C.

Note that a classical compact Vuza canon is a modulus p compact Vuza canon.

We can compute very quickly Vuza RT_2C, while classical Vuza canons are extremely long to compute. The link between classical Vuza canon and Vuza RT_2C is a hope for mathematicians to find a way to compute them faster, namely to go from modulus p canons to classical canons.

In order to understand this path, we are from now on going to focus on what makes a difference between a classical canon and a modulus p canon: the superposition of onsets. Such layered onsets will be called 'donsets', and their set will be denoted with D.

Definition 5: If (A, B) is a RT_pC of $\mathbb{Z}_N$, the donsets set is

$$D = \bigcup_{n \in \mathbb{Z}_N} \bigcup_{j=1}^{k} \{n, \mathbb{1}_{(A+B)}(n) = kp + 1\}.$$

Example 5: The donsets set of example 3 is $\{3\}$, as clearly shown in figure 2.

Note that if (A, B) is a RT_pC with $D = \emptyset$, then it is a classical RTC, and it cannot be a Vuza canon because of property 2. For instance, it is the case of the second canon in example 4. This is another reason to focus on RT_pC with non-empty donsets set.

It is musically very interesting to understand given a RT_pC where its donsets will appear. Actually, the beats where the composer get layered onsets are the strongest ones, where they can express harmony; or a superposition of events, if the onsets represent actions a musician has to execute at each time. It is simply what makes the strength of RT_pC, but also its interest from a mathematical point of view. As a matter of fact, if one can understand the direct link between a rhythmic pattern and the generated donsets, one can get the chance to link back RT_pC to classical RTC.

Nevertheless, one can easily see (for instance under the duality operation, consisting of reversing the roles of inner and outer rhythms) that there is not uniqueness in this link. It means that given a donsets set one may find many RT_pC that generate it. However, those RT_pC are often associated by a transformation, as we will see in the next section.

5. TRANSFORMATIONS GENERATING IDENTICAL DONSETS

Some new theorems presented in this section give transformations of RT_2C that generate bigger RT_2C with the same donsets, like duality would obviously do. All these transformations have been implemented in OpenMusic.

5.1 Zooming

Theorem 2: If (A, B) is a RT_2C of $\mathbb{Z}_N$, for all $k \in \mathbb{N}^*$, both $(A_k, \widetilde{B_k})$ and $(\widetilde{A_k}, B_k)$ are RT_2C of $\mathbb{Z}_{kN}$, and they have the same donsets, with (same for B)

$$A_k = \{ka, a \in A\}$$

$$\widetilde{A_k} = \{ka, ka+1, \ldots, ka + (k-1), a \in A\}$$

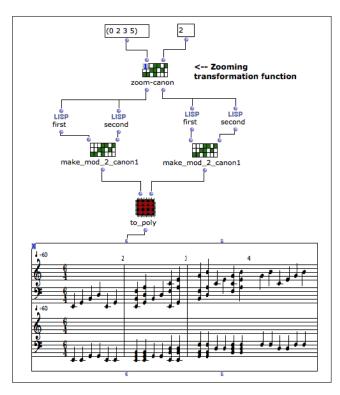See the patch generating the zoom of a canon presented in figure 4.

Figure 4. An OpenMusic patch returning two greater canons with the same donsets from a canon and a zooming coefficient, then representing them.

5.2 Transferring

Lemma 1: Let $1 < p < N$ a divisor or N, then

$$\left(\left\{ 0, p, 2p, \ldots, (\frac{N}{p} - 1)p \right\}, \{0, 1, 2, \ldots, p-1\} \right)$$

tiles $\mathbb{Z}_N$

Theorem 3: If (A, B) is a RT$_2$C of $\mathbb{Z}_N$, and $1 < p < N$ a divisor or N, then both $(A_p, \widetilde{B_p})$ and $(\widetilde{A_p}, B_p)$ are RT$_2$C of $\mathbb{Z}_{2N}$, and they have the same donsets with (same for B)

$$A_p = A + \{0, p\}$$

$$\widetilde{A_p} = A + \left\{ 0, p, 2p, \ldots, (\frac{N}{p} - 1)p \right\}$$

if and only if $\widetilde{A_p}$, as well as $\widetilde{B_p}$, verifies

$$\mathbb{1}_{\widetilde{A_p}}(n) > 1 \Rightarrow n = 0 \bmod (p).$$

The transferring operation is illustrated in the OpenMusic patch presented in figure 5 which provides the following informations given an inner rhythm A:

- it provides all primes p dividing N the period of the compact RT$_2$C (A, B),

- it verifies if both A and B verify the condition of theorem 3 for some of those p and returns them,

  (`is_transferable_p`)

- it then applies the transferring transformation to the canon with one of those p and returns the two new bigger RT$_2$C having the same donsets.
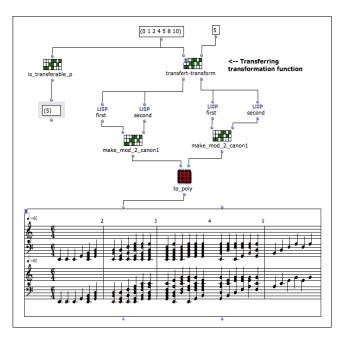
  (`transfert-transform`)

Figure 5. An OpenMusic patch returning two greater canons with the same donsets from a canon verifying the transferring condition.

5.3 Musical and mathematical interest of these transformations and some perspectives

Starting from a given canon, the composer can now generate a family of canons having the same donsets. These canons can be played together at the same speed creating in such a way a climax on the beats with layered onsets. Different notes can be affected to the different voices, and the donsets will be heard as sudden wide chords opposed to the linear character of the melodies.

Mathematically, knowing that such transformations exist is relevant in many ways. First, given a rhythmic pattern it is pretty quick to find out if it will produce a Vuza RT$_2$C. On the contrary to obtain all Vuza of a given period is exponential because we still have to exhaustively produce all possible rhythmic patterns and test them. Hence, those transformations provide a way to obtain Vuza canons of a great period (having furthermore some important information about cardinality of A, B and D), and are the first steps of maybe finding all generating transformations. This would reduce the exponentially long search of all modulus p Vuza canon of a given period to a problem running in polynomial time.

Yet more importantly, the transformations we have presented earlier not only generate greater canons, but these resulting canons have all the same donsets. It would be even more capital to find all transformations with this property than all possible transformations. Indeed, if we can find them all, we could quotient the set of Vuza canons by those transformations, and then find a bijection between donsets sets and this quotient set, allowing us to go from modulus p canons to classical canons!

But finding those transformations will not be an easy task. For instance, here are the three RT$_2$C of $\mathbb{Z}_{30}$ depicted in figure 6 which have the same donsets:
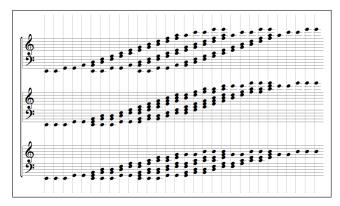
Figure 6. Three compact RT_2C of $\mathbb{Z}_{30}$ with the same donsets.

1. $(\{0, 1, 5, 6, 10, 11\},$
   $\{0, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16, 18\})$

2. $(\{0, 1, 2, 5, 6, 7\},$
   $\{0, 3, 5, 6, 8, 9, 10, 11, 12, 13, 14, 16, 17, 19, 22\})$

3. $(\{0, 1, 2, 5, 6, 7, 10, 11, 12\},$
   $\{0, 3, 5, 6, 8, 9, 11, 12, 14, 17\})$

The two last ones come from the 5-transferring transformation of the RT_2C $(\{0, 1, 2\}, \{0, 3, 6, 9, 12\})$ of $\mathbb{Z}_{15}$.

The first one looks like it comes from the 5-transferring of the canon $(\{0, 1\}, \{0, 2, 4, 6, 8, 10, "12/14"\})$ of $\mathbb{Z}_{15}$. Indeed, 15 is not divisible by 2, so $\{0, 1\}$ cannot tile $\mathbb{Z}_{15}$ (with the "12" it tiles $\mathbb{Z}_{14}$ and with the "14" it tiles $\mathbb{Z}_{16}$).

Hence, the transferring transformation is applicable to a larger range of canons, and theorem 3 have to be extended to take into account those canons that "could exist". The first canon of the example is indeed twice the size of the original canon, and has the same donsets as both the other ones. It satisfies the same properties as if it is obtained from the 5-transferring

$(\{0, 1\} + \{0, 5, 10\}, \{0, 2, 4, 6, 8, 10, "12/14"\} + \{0, 5\})$.

This special case makes us think that transferring transformation may be more widely used, and moreover, some other transformations generating canons with the same donsets may exist.

6. CONCLUSION

Finding a way to obtain all the Vuza RTC of a given period $\mathbb{Z}_N$ is an issue for composers, but also for mathematicians, because finding a necessary and sufficient condition to know if a rhythmic pattern will tile the line is equivalent to solve the spectral conjecture in dimension 1. Tao has recently proved this conjecture, also called Fuglede conjecture, to be false for dimensions higher or equal to 5 [15], and from this work, Matolcsi has proved it to be false in dimension 4 [16], and Farkas in dimension 3 [17]. The problem remains open in dimensions 1 and 2. Dimension 1 is precisely the case which concerns the tiling canons constructions.

Composers have been using classical RTC for years (see [8] and [9]), and we hope that the implementation of these new tools in OpenMusic will make musicians aware of the existence of RT_pC and the musical potentials they have.

RT_pC are a very recent field of study, and there is still so much to be done to understand them. We already have some cardinality results for specials cases ([14]), and we need to use some mathematical theorems to make their implementation faster. For instance, one can prove that we can never find a RT_2C that is not a RTC for the period N in the sequence A071642 of Sloanes OEIS [18]. We also hope that this article will bring this subject not only to the eyes of composers and musicians, but also to the community of computational musicologists and computer scientists.

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